HOPF ALGEBRA STRUCTURE OF A MODEL QUANTUM FIELD THEORY

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ABSTRACT. Recent elegant work[1] on the structure of Perturbative Quantum Field Theory (PQFT) has revealed an astonishing interplay between analysis(Riemann Zeta functions), topology (Knot theory), combinatorial graph theory (Feynman Diagrams) and algebra (Hopf structure). The difficulty inherent in the complexities of a fully-fledged field theory such as PQFT means that the essential beauty of the relationships between these areas can be somewhat obscured. Our intention is to display some, although not all, of these structures in the context of a simple zero-dimensional field theory; i.e. a quantum theory of non-commuting operators which do not depend on space-time. The combinatorial properties of these boson creation and annihilation operators, which is our chosen example, may be described by graphs [2, 3], analogous to the Feynman diagrams of PQFT, which we show possess a Hopf algebra structure[4]. Our approach is based on the partition function for a boson gas. In a subsequent note in these Proceedings we sketch the relationship between the Hopf algebra of our simple model and that of the PQFT algebra.

1. PARTITION FUNCTION INTEGRAND

Consider the Partition Function Z of a quantum statistical mechanical system

$$(1.1) \qquad Z = \operatorname{Tr} \exp(-\beta H).$$

whose hamiltonian is H ($\beta \equiv 1/kT$, k=Boltzmann's constant T=absolute temperature). We may evaluate the trace over any complete set of states; we choose the (over-)complete set of coherent states

$$(1.2) \qquad |z\rangle = e^{-|z|^2/2} \sum_n (z^n/n!) a^{\dagger n} |0\rangle$$

where $a^\dagger$ is the boson creation operator satisfying $[a.a^\dagger] = 1$, for which the completeness or *resolution of unity* property is

$$(1.3) \qquad \frac{1}{\pi} \int d^2z |z\rangle\langle z| = I \equiv \int d\mu(z)|z\rangle\langle z|.$$

The simplest, and generic, example is the free single-boson hamiltonian $H = \epsilon a^\dagger a$ for which the appropriate trace calculation is

$$
\begin{aligned}
Z &= \frac{1}{\pi} \int d^2z \langle z| \exp\bigl(-\beta\epsilon a^\dagger a\bigr)|z\rangle = \\
(1.4) \qquad &= \frac{1}{\pi} \int d^2z \langle z| : \exp\bigl(a^\dagger a(e^{-\beta\epsilon} - 1)\bigr) : |z\rangle.
\end{aligned}
$$

Here we have used the well-known relation [5, 6] for the *forgetful* normal ordering operator : $f(a, a^\dagger)$: which means "normally order the creation and annihilation operators in f *forgetting* the commutation relation $[a, a^\dagger] = 1$."[1]

We may write the Partition Function in general as

$$(1.5) \qquad Z = \int F(x, z)\, d\mu(z); \quad (x \equiv -\beta\epsilon)$$

thereby defining the Partition Function Integrand (PFI) $F(x, z)$, which will be the object of our analysis.

2. Combinatorial aspects: Bell numbers

The generic free-boson example Eq.(1.4) above may be rewritten to show the connection with certain well-known combinatorial numbers. Writing $y = |z|^2$, Eq.(1.4) becomes

$$(2.1) \qquad Z = \int\limits_0^\infty dy \exp\big(y(e^x - 1)\big).$$

This is an integral over the classical *exponential generating function* for the *Bell polynomials*

$$(2.2) \qquad \exp\big(y(e^x - 1)\big) = \sum_{n=0}^\infty B_n(y)\, \frac{x^n}{n!}$$

where the Bell number is $B_n(1) = B(n)$, the number of ways of putting n different objects into n identical containers (some may be left empty). Related to the Bell numbers are the *Stirling numbers of the second kind $S(n, k)$*, which are defined as the number of ways of putting n different objects into k identical containers, leaving none empty. From the definition we have $B(n) = \sum_{k=1}^n S(n, k)$. The foregoing gives a combinatorial interpretation of the partition function integrand $F(x, y)$ as the exponential generating function of the Bell polynomials.

2.1. **Graphs.** We now give a graphical representation of the Bell numbers. Consider labelled lines which emanate from a white dot, the origin, and finish on a black dot, the vertex. We shall allow only one line from each white dot but impose no limit on the number of lines ending on a black dot. Clearly this simulates the definition of $S(n, k)$ and $B(n)$, with the white dots playing the role of the distinguishable objects, whence the lines are labelled, and the black dots that of the indistinguishable containers. The identification of the graphs for 1, 2 and 3 lines is given in Figure 1.

We have concentrated on the Bell number sequence and its associated graphs since, as we shall show, there is a sense in which this sequence of graphs is *generic*. That is, we can represent *any* combinatorial sequence by the same sequence of graphs as in Figure 1, with suitable vertex multipliers (denoted by the V terms in the same figure). Consider the general partition function of Eq.(1.1) with hamiltonian given by $H = \epsilon w(a, a^\dagger)$, with w a string (= sum of products of positive powers)

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[1]Of course, this procedure may alter the value of the operator to which it is applied.
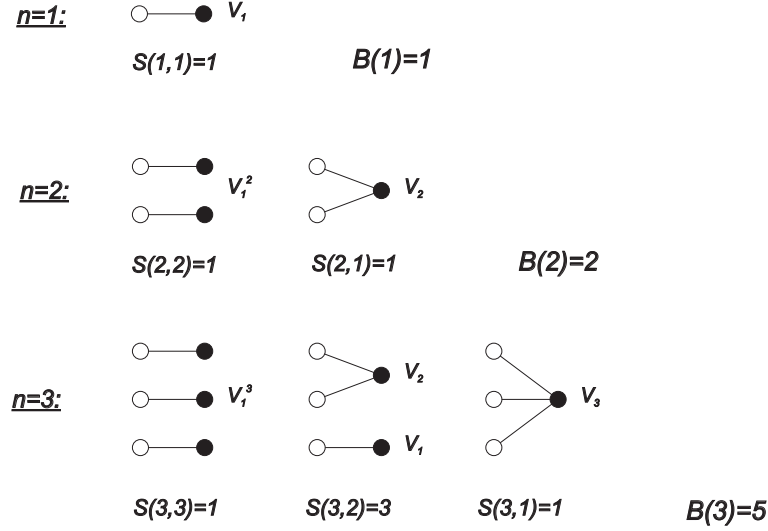
FIGURE 1. Graphs for $B(n)$, $n = 1, 2, 3$.

of boson creation and annihilation operators. The partition function integrand F of Eq.(1.5) for which we seek to give a graphical expansion, is now

$$
\begin{aligned}
F(x, z) &= \langle z| \exp(xw)|z\rangle = & (x = -\beta\epsilon) \\
&= \sum_{n=0}^{\infty} \langle z|w^n|z\rangle \frac{x^n}{n!} = \\
&= \sum_{n=0}^{\infty} W_n(z) \frac{x^n}{n!} = \\
&= \exp\left(\sum_{n=1}^{\infty} V_n(z) \frac{x^n}{n!} \right),
\end{aligned}
$$

(2.3)

with obvious definitions of W_n and V_n. The sequences $\{W_n\}$ and $\{V_n\}$ may each be recursively obtained from the other [7]. This relates the sequence of multipliers $\{V_n\}$ of Figure 1 to the hamiltonian of Eq.(1.1). The lower limit 1 in the V_n summation is a consequence of the normalization of the coherent state $|z\rangle$. A mild generalization is to write the Partition Function Integrand of Eq.(2.3), using the product formula [3, 8, 9], as

(2.4) $$F(x, \mathbb{V}, \mathbb{L}) = \exp\left(\sum_{m=1}^{\infty} L_m \frac{x^m}{m!} \frac{d^m}{dy^m} \right) \exp\left(\sum_{s=1}^{\infty} V_s \frac{y^s}{s!} \right)\Bigg|_{y=0}.$$

(We have suppressed the explicit dependence on the coherent state parameter z.) The advantage of this formulation is that it treats the white and black spots symmetrically[2], as well as having some calculational advantages. An example of some of the associated graphs is given in Figure 2.

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[2]Thus we need not adhere to the previous convention of treating the white spots as the origins.

3. HOPF ALGEBRA STRUCTURE

We briefly describe the Hopf algebra **BELL** which the diagrams of Figure 1 define.

(1) Each distinct diagram is an individual basis element of **BELL**; thus the dimension is infinite. (Visualise each diagram in a "box".) The sum of two diagrams is simply the two boxes containing the diagrams. Scalar multiples are formal; for example, they may be provided by the V coefficients.

(2) The identity element e is the empty diagram (an empty box).

(3) Multiplication is the juxtaposition of two diagrams within the same "box". **BELL** is generated by the *connected* diagrams; this is a consequence of the Connected Graph Theorem [10]. Since we have not here specified an order for the juxtaposition, multiplication is commutative.

(4) The coproduct $\Delta :$ **BELL** $\longrightarrow$ **BELL** $\times$ **BELL** is defined by

$$\begin{aligned} \Delta(e) &= e \times e \quad (\text{unit } e) \\ \Delta(x) &= x \times e + e \times x \quad (\text{generator } x) \\ \Delta(AB) &= \Delta(A)\Delta(B) \quad \text{otherwise} \end{aligned}$$

so that Δ is an algebra homomorphism.

(5) The co-unit ϵ satisfies $\epsilon(e) = 1$ otherwise $\epsilon(A) = 0$.

(6) The antipode $\mathcal{S} :$ **BELL** $\longrightarrow$ **BELL** satisfies $\mathcal{S}(e) = e$; on a generator x, $\mathcal{S}(x) = -x$. It is an *anti-homomorphism*, i.e. $\mathcal{S}(AB) = \mathcal{S}(B)\mathcal{S}(A)$.

It may be shown that the foregoing structure **BELL** satisfies the axioms of a commutative, co-commutative Hopf algebra. Diagrams such as those of Figure 2, associated with the formulation Eq.(2.4) similarly give rise to a commutative, co-commutative Hopf algebra **DIAG** generated by the connected graphs. The Bell Hopf algebra **BELL** is a homomorphic image of **DIAG**.
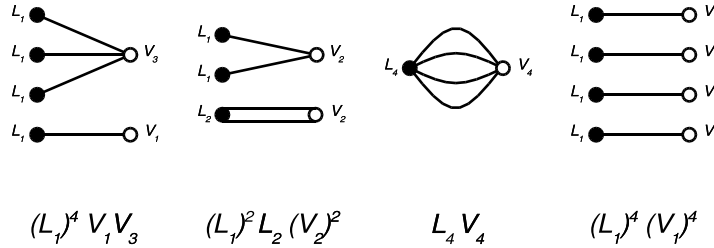

FIGURE 2. Some examples of 4-line graphs.

4. Discussion

The philosophy of our approach has been the following. To elucidate the structure of a complicated physical system, such as perturbative quantum field theory (PQFT) , is itself a daunting task. One may gain some insight by considering a much more straightforward system, such as the one treated here. This may be thought of as a zero-dimensional field theory, as our simple boson operator a does not depend on space or time. Nevertheless, we have shown that even such a basic system does exhibit some of the features of the more complicated case, in particular the structure of the Hopf algebras **BELL** and **DIAG**. This may be thought of as a simple solvable model in its own right. However, one may also adopt the approach of asking wherein does this simple structure sit within the full PQFT structure? Our approach is to generalize the *algebraic* structure, and thereby produce Hopf algebras of sufficient complexity to emulate those associated with PQFT.

In the following note [11] we show how such a generalization may be achieved, with starting point the Hopf algebras **BELL** and **DIAG** described here.

References

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